EXCILL V: EXTREMAL COMBINATORICS AT ILLINOIS SUMMER SCHOOL ON DESIGN THEORY WORKSHEET

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Here is a collection of exercises that were either introduced during the lectures, or are provided as supplementary practice. We do not expect all problems can be completed in an hour, so we recommend picking the problems that interest you most.

1. Counting designs

Two order-n Latin squares L and L' are *orthogonal* if no two cells contain the same combination of entry from L and entry from L'. That is, $\{(L_{i,j}, L'_{i,j}) : i, j \in [n]\} = [n] \times [n]$. Let $\mathcal{O}_n$ denote the set of pairs of orthogonal Latin squares of order-n using symbol set $[n]$.

The question of the existence of orthogonal Latin squares dates back to Euler. He proposed the famous "thirty-six officers problem":

> A very curious question, which has exercised for some time the ingenuity of many people, has involved me in the following studies, which seem to open a new field of analysis, in particular the study of combinations. The question revolves around arranging 36 officers to be drawn from 6 different regiments so that they are ranged in a square so that in each line (both horizontal and vertical) there are 6 officers of different ranks and different regiments. –Leonhard Euler

Euler also conjectured that orthogonal Latin squares of order-n exist if and only if $n \not\equiv 2 \mod 4$. However, this was disproved by Bose, Shrikhande, and Parker in 1959. It turns out that orthogonal Latin squares exist for every order except for 2 and 6. (For $n = 4$, the problem yields a fun puzzle on arranging the Jacks, Queens, Kings, and Aces of a deck of cards in a 4×4 grid such that both the ranks and the suits form a Latin square.) In this problem, you will show that for all large n, we can fairly precisely determine the number of pairs of orthogonal Latin squares. Euler would be surprised!

Problem 1. Let $n \in \mathbb{N}$.

(a) Construct a graph H_n, a graph F, and a bijection from the set of F-decompositions of H_n to $\mathcal{O}_n$.

(b) Construct a uniform hypergraph $\mathcal{H}_n$ and a bijection from the set of perfect matchings of $\mathcal{H}_n$ to $\mathcal{O}_n$.

(c) Compute the degrees of vertices and pairs of vertices and the uniformity of this hypergraph, and explain why

$$|\mathcal{O}_n| = \left((1 \pm o(1))\frac{n^2}{e^5}\right)^{n^2}.$$

2. The absorber-template method

This problem is the exercise from Lecture 2.

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Problem 2. Generalize the approach presented in Lecture 2 to prove the following: for all $k \geq 3$ and $\gamma > 0$, if $p \geq 1/n^{k/2-\gamma}$, then a.a.s. $\mathcal{H}^{(k)}_{n,p}$ contains a perfect matching, provided $k \mid n$. Explain why the approach fails for $p \leq 1/n^{k/2+\gamma}$

3. Absorbers and boosters

Recall the definition of absorber. We will also need *transformers* for the next problem.

Definition 3.1. An F-*absorber* for a graph L is a graph A such that $V(L) \subseteq V(A)$ is independent in A and both A and $A \cup L$ have an F-decomposition, and an $(H, H'; F)$-*transformer* is a graph T such that $V(H) \cup V(H') \subseteq V(T)$ is independent in T and both $T \cup H$ and $T \cup H'$ have an F-decomposition.

Definition 3.2. For a graph H and a subset $U \subseteq V(H)$, the *degeneracy* of H rooted at U is the smallest $d \in \mathbb{N}$ such that there exists an ordering $v_1, \ldots, v_{|V(H)|-|U|}$ of the vertices of $V(H) \setminus U$ such that for all $i \in [|V(H)| - |U|]$, the vertex v_i has at most d neighbors in $U \cup \{v_j : j \in [i-1]\}$.

Suppose A is an absorber for a graph L with degeneracy rooted at $V(L)$ at most d. Note that if $1/n \ll \varepsilon, 1/d < 1$ and G is an n-vertex graph satisfying $\delta(G) \geq (1 - 1/d + \varepsilon)n$, then for every copy of L in G, there is a copy of $A \cup L$ in G. This fact motivates finding absorbers with low rooted degeneracy, as in the following problem.

Problem 3. Let $k \geq 3$ be an integer, and let L be a graph such that $2 \mid d_L(v)$ for every $v \in V(L)$.

(a) Prove that if $L' \cong C_{|E(L)|}$ is disjoint from L, then there is an $(L, L'; C_{2k})$-transformer with degeneracy rooted at $V(L) \cup V(L')$ at most two. *Hint: Use that L is Eulerian.*

(b) Prove that if $2k \mid |E(L)|$, then there is a C_{2k}-absorber for L with degeneracy rooted at $V(L)$ at most two.

Recall the following definition of a booster and rooted density.

Definition 3.3. Let $q > 1$ be an integer.

- A K_q-*booster* is a graph B along with two K_q-decompositions $\mathcal{B}_1$ and $\mathcal{B}_2$ of B such that $\mathcal{B}_1 \cap \mathcal{B}_2 = \varnothing$.
- A tuple $(B, \mathcal{B}_1, \mathcal{B}_2, S)$ is a *rooted K_q-booster* if
  - B is a graph,
  - $\mathcal{B}_2$ is a K_q-decomposition of B, called the *off-decomposition*,
  - $S \cong K_q$, $V(S) \subseteq V(B)$, and S is edge-disjoint from B,
  - $\mathcal{B}_1$ is a K_q-decomposition of $B \cup S$ with $S \notin \mathcal{B}_1$, called the *on-decomposition*, and
  - $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$.
- For a nonempty K_q-packing $\mathcal{H}$ of a graph G and $R \subseteq V(G)$, the *rooted density* of $\mathcal{H}$ and R is
$$d(\mathcal{H}, R) := |\mathcal{H}| \Bigg/ \left| \bigcup_{H \in \mathcal{H}} V(H) \setminus R \right|,$$
and the *maximum rooted density* of $\mathcal{H}$ and R is
$$m(\mathcal{H}, R) := \max_{\mathcal{H}' \subseteq \mathcal{H}} d(\mathcal{H}', R).$$
- For a rooted K_q-booster $(B, \mathcal{B}_1, \mathcal{B}_2, S)$, the *rooted density* is $\max_{\mathcal{H} \in \{\mathcal{B}_1, \mathcal{B}_2\}} m(\mathcal{H}, V(S))$.

Problem 4. Prove that for every $\varepsilon > 0$, there exists a rooted K_3-booster with rooted density at most $1 + \varepsilon$.

Sah, Sawhney, and Simkin [Threshold for Steiner triple systems] used these boosters to construct a $(1/n^{1+o(1)})$-spread measure on order-n Steiner triple systems.

4. Working with spread measures

These problems are from Lecture 4. Recall the following definition.

Definition 4.1. Let $\mathcal{F} \subseteq \binom{E(K_n^{(k)})}{\ell}$ be a set of n-vertex ℓ-edge k-uniform hypergraphs. A probability measure μ on $\mathcal{F}$ is *σ-spread* if

$$(\star) \qquad \mathbb{P}_{H\sim\mu}[H \supseteq S] \leq \sigma^{|S|} \qquad \text{for all } S \subseteq E(K_n^{(k)}).$$

Problem 5. Prove that the uniform distribution on perfect matchings in $K_n^{(k)}$ is

$$\left(((k-1)! + o(1))\frac{e^{k-1}}{n^{k-1}}\right)\text{-spread.}$$

Problem 6. Prove that for every $k, \Delta > 0$, there exists $C > 0$ such that the following holds for all sufficiently large n. If H is a k-uniform n-vertex hypergraph with maximum degree at most Δ, then the uniform distribution on copies of H in $K_n^{(k)}$ is

$$\left(\frac{C}{n^{1/m_1(H)}}\right)\text{-spread,}$$

where $m_1(H) = \max_{H' \subseteq H} \frac{|E(H')|}{|V(H')|-1}$.

For the next problem, you need the conditional Lovász Local Lemma.

Lemma 4.2. *Let $\{X_i\}_{i\in I}$ be independent random variables, and let $\{\mathcal{E}_j\}_{j\in J}$ be events where each $\mathcal{E}_j$ depends on a subset $S_j \subseteq I$ of the variables. Let Γ be a graph with vertex set J where $S_j \cap S_{j'} = \emptyset$ for all distinct non-adjacent $j, j' \in J$. Let $\mathbb{P}$ be the product measure on the random variables $\{X_i\}_{i\in I}$, and let $\mathcal{E}$ be an event depending on a subset $S \subseteq I$. If $\mathbb{P}[\mathcal{E}_j] \leq p$ for all $j \in J$ and $4p\Delta(\Gamma) \leq 1$, then*

$$\mathbb{P}\left[\mathcal{E} \,\middle|\, \bigcap_{j\in J} \overline{\mathcal{E}_j}\right] \leq \mathbb{P}[\mathcal{E}] \exp(6pN),$$

where N is the number of events $\mathcal{E}_j$ with $S_j \cap S \neq \emptyset$.

Problem 7. Prove that for every $k \geq 2$, there exists a constant $C \geq 1$ such that the following holds. If $\mathcal{G}$ is a bipartite k-uniform hypergraph with bipartition (A, B), and if there exists $D \geq 1$ such that

- every vertex of A has degree at least $8kD$ in $\mathcal{G}$ and
- every vertex of B has degree at most D in $\mathcal{G}$,

then there is a (C/D)-spread distribution on A-perfect matchings of $\mathcal{G}$.