EXCILL V CONFERENCE: SUMMER SCHOOL ON DESIGN THEORY LECTURE 4. THRESHOLDS FOR DESIGNS VIA SPREADNESS AND REFINED ABSORPTION

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1. Introduction

The goal of this lecture is to explain how the absorber-template method from the previous lecture can be used to upper bound the threshold for designs. The main ingredients are:

- constructing a "well-spread measure" on designs yields an upper bound on the threshold via the Park–Pham Theorem;
- refined absorption, introduced by Delcourt and Postle, can be viewed as the "high-dimensional" version of the absorber-template method from the previous lecture.

1.1. **Introduction to designs.** Some of the content in this subsection will be review and may be skipped. Recall that, for $n > q > r \geq 1$, an (n, q, r)-*Steiner system* is an n-vertex q-uniform hypergraph where every r-set of vertices is in exactly one edge.

Example 1.1.

- An $(n, q, 1)$-Steiner system is a q-uniform perfect matching.
- An $(n, 3, 2)$-Steiner system is a Steiner triple system of order n.

Definition 1.2. For hypergraphs F and G,

- an F-*packing* of G is a collection of pairwise edge-disjoint copies of F in G, and
- an F-*decomposition* of G is an F-packing covering every edge of G.

Observation 1.3. *(n, q, r)-Steiner systems are in bijection with $K_q^{(r)}$-decompositions of $K_n^{(r)}$*

Observation 1.4 (The divisibility conditions)**.** *If an (n, q, r)-Steiner system exists, then*

$$(\dagger) \qquad \binom{q-i}{r-i} \mid \binom{n-i}{r-i} \qquad \textit{for every } 0 \leq i \leq r-1.$$

The "Existence Conjecture" for designs, dating back to the 1850s, asserted that for every $q > r \geq 1$ and n sufficiently large, if ($\dagger$) holds, then there exists an (n, q, r)-Steiner system.

- The conjecture is trivial for $r = 1$.
- Kirkman [15] proved the case $q = 3$ and $r = 2$ in 1847. This case corresponds to triangle decompositions of K_n and Steiner triple systems.
- Wilson [19, 20, 21] proved the $r = 2$ case in the 1970s, which corresponds to K_q decompositions of K_n.
- Rödl [17] proved in 1985 that "almost" (n, q, r)-Steiner systems exist, using what we now call the nibble method.
- Keevash [13] in 2014 proved the full conjecture using "randomized algebraic construction".
- Glock, Kühn, Lo, and Osthus [8] gave a new proof in 2016 using "iterative absorption".
- Delcourt and Postle [5] in 2024 gave a third proof using "refined absorption".

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1.2. **The threshold conjecture.** We are interested in the random analogue. Let $\mathcal{H}^{(q)}_{n,p}$ be the binomial random q-uniform hypergraph. What is the threshold for $\mathcal{H}^{(q)}_{n,p}$ to contain an (n, q, r)-Steiner system? There is an obvious lower bound of $n^{-q+r} \log n$, which is the threshold for every r-set to be contained in at least one edge. Kang, Kelly, Kühn, Methuku, and Osthus [12, Conjecture 1.14] proposed a matching upper bound.

Conjecture 1.5 (The threshold for designs conjecture; Kang, Kelly, Kühn, Methuku, and Osthus [12])**.** *For every $q, r \in \mathbb{N}$ with $q > r$, the following holds. If (†) holds and $p = \omega(n^{-q+r} \log n)$, then a.a.s. $\mathcal{H}^{(q)}_{n,p}$ contains an (n, q, r)-Steiner system.*

The $r = 1$ case is Shamir's problem – the threshold for a perfect matching in a random q-uniform hypergraph – which we studied in the previous lecture. As discussed in the previous lecture, the threshold is known to be $\log n/n^{q-1}$, which is the same as the threshold for having no isolated vertices.

The only other case that is known is the $q = 3$ and $r = 2$ case, which corresponds to Steiner triple systems. Following the breakthroughs of Frankston, Kahn, Narayanan, and Park [7] and Park and Pham [16] on the Kahn–Kalai Conjecture [11], a series of results in 2022 culminated in the resolution of this case:

- first, Sah, Sawhney, and Simkin [18] proved an upper bound on the threshold of $n^{-1+o(1)}$;
- Kang, Kelly, Kühn, Methuku, and Osthus [12] proved the better bound of $n^{-1} \log^2 n$;
- subsequently Jain and Pham [10] and independently Keevash [14] settled the problem, proving that $n^{-1} \log n$ upper bounds the threshold. In fact, Jain and Pham [10] and Keevash [14] first proved that $n^{-1} \log n$ upper bounds the Latin square threshold and used a reduction of Kang, Kelly, Kühn, Methuku, and Osthus [12] to then derive the Steiner triple system threshold.

All these proofs bound the threshold by constructing a "well-spread" probability measure. We will see how to use this approach to prove the following result.

Theorem 1.6 (Delcourt, Kelly, and Postle [4])**.** *For every integer $q > 2$, if n satisfies the necessary divisibility conditions and $p \geq n^{-(q-6)/2}$, then a.a.s. $\mathcal{H}^{(q)}_{n,p}$ contains an $(n, q, 2)$-Steiner system.*

By covering this proof, we will also see how to use the method of *refined absorption*, introduced by Delcourt and Postle [5].

2. Spreadness and Park–Pham

The Park–Pham theorem gives a general way to prove threshold upper bounds from spread distributions. We will only introduce a special case of the theorem.

Definition 2.1. Let $\mathcal{F} \subseteq \binom{E(K_n^{(k)})}{\ell}$ be a set of n-vertex ℓ-edge k-uniform hypergraphs. A probability measure μ on $\mathcal{F}$ is *σ-spread* if

$$(\star) \qquad \mathbb{P}_{H\sim\mu}\left[H \supseteq S\right] \leq \sigma^{|S|} \qquad \text{for all } S \subseteq E(K_n^{(k)}).$$

Every measure is trivially 1-spread, so we are interested in minimizing σ. To get a feel for the notion of spread, it is useful to consider the two extremes for $|S|$. If $H \sim \mu$ where μ is σ-spread, then

- no edge appears in H with probability more than σ, and
- since $1 = \sum_{F\in\mathcal{F}} \mathbb{P}\left[H = F\right] \leq |\mathcal{F}|\sigma^{\ell}$, the family $\mathcal{F}$ has size at least $\sigma^{-\ell}$.

Note also that if each edge is included in H with probability at most σ and these events are not positively correlated, then ($\star$) holds. Hence, ($\star$) can also be viewed as saying that the events that individual edges appear in H are not too strongly correlated.

By Linear Programming duality, the minimum σ for which $\mathcal{F}$ supports a σ-spread measure is roughly the *fractional expectation threshold* for $\mathcal{F}$. Although we will not define it formally, it is important to note that the fractional expectation threshold is a lower bound on the threshold for $\mathcal{H}_{n,p}^{(k)}$ to contain a subgraph in $\mathcal{F}$. In fact, the fractional expectation threshold can roughly be thought of as, "the best lower bound on the threshold that could be proved with the first moment method".

Frankston, Kahn, Narayanan, and Park [7] proved the "fractional version" of the Kahn–Kalai conjecture, that the threshold is always at most a logarithmic factor greater than the fractional expectation threshold.

Theorem 2.2 (Special case of Park–Pham [16] and FKNP [7])**.** *There exists a constant K such that the following holds. If $\mathcal{F} \subseteq \binom{E(K_n^{(k)})}{\ell}$ supports a σ-spread measure and $p \geq K\sigma \log \ell$, then a.a.s. (as $\ell \to \infty$) $\mathcal{H}_{n,p}^{(k)}$ contains some $F \in \mathcal{F}$.*

To see an example of this theorem, let us use it to prove that the threshold for $G(2n, p)$ to contain a perfect matching is at most $\log n/n$. It will follow immediately from Theorem 2.2 and the following lemma.

Lemma 2.3. *The uniform distribution on perfect matchings in $K_{n,n}$ is (e/n)-spread.*

Proof. Let M be a perfect matching chosen uniformly at random in $K_{n,n}$, and let $e_1, \ldots, e_s \in E(K_{n,n})$. We want to show $\mathbb{P}[M \ni e_1, \ldots, e_s] \leq (e/n)^s$. We can assume $\{e_1, \ldots, e_s\}$ is a matching, or else $\mathbb{P}[M \ni e_1, \ldots, e_s] = 0$, as desired. Hence,

$$\mathbb{P}[M \ni e_1, \ldots, e_s] = \frac{(n-s)!}{n!} \leq \left(\frac{1}{n}\right)\left(\frac{1}{n-1}\right)\cdots\left(\frac{1}{n-s+1}\right) \leq \left(\frac{e}{n}\right)^s,$$

as desired. □

A similar approach yields the following.

Exercise 2.4. *The uniform distribution on perfect matchings in $K_n^{(k)}$ is*

$$\left(((k-1)! + o(1))\frac{e^{k-1}}{n^{k-1}}\right)\text{-spread.}$$

Exercise 2.4 and Theorem 2.2 together yield the solution to Shamir's problem. Besides Shamir's problem, one of the other major motivations for the Kahn–Kalai conjecture was the threshold for $G(n, p)$ to contain a copy of some fixed bounded-degree spanning tree. This result follows from Theorem 2.2 combined with the following exercise.

Exercise 2.5. *For every $k, \Delta > 0$, there exists $C > 0$ such that the following holds for all sufficiently large n. If H is a k-uniform n-vertex hypergraph with maximum degree at most Δ, then the uniform distribution on copies of H in $K_n^{(k)}$ is*

$$\left(\frac{C}{n^{1/m_1(H)}}\right)\text{-spread,}$$

where $m_1(H) = \max_{H' \subseteq H} \frac{|E(H')|}{|V(H')|-1}$.

For designs, it is natural to believe that the uniform distribution is also sufficiently spread to yield Conjecture 1.5.

Conjecture 2.6 (The spreadness of designs conjecture)**.** *The uniform distribution on (n, q, r)-Steiner systems is*

$$\left((1 + o(1)) \exp\left(\binom{q}{r} - 1\right) \Big/ \binom{n-r}{q-r}\right)\text{-spread.}$$

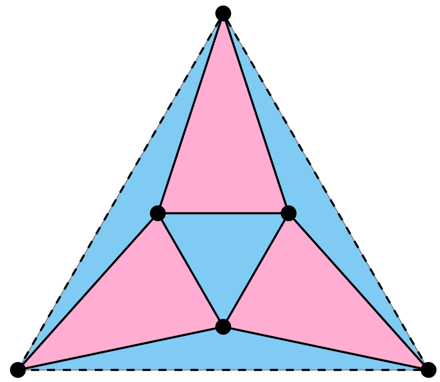

FIGURE 1. A K_3-absorber for the outer triangle

To prove Conjecture 1.5 with Theorem 2.2, it suffices to find a (not necessarily uniform) distribution that is $O(1/n^{q-r})$-spread, so Conjecture 2.6 is stronger than necessary in two respects: it posits that the uniform distribution witnesses small spread, and it determines the leading constant factor. Note that ($\star$) holds for $\sigma = \left((1+o(1))\exp\left(\binom{q}{r}-1\right) \Big/ \binom{n-r}{q-r}\right)$ when $|S| = 1$, by symmetry, and when $|S| = \binom{n}{r} / \binom{q}{r}$ by the enumeration results presented in the previous lecture. However, Conjecture 2.6 is wide open even for Steiner triple systems, and the analogous conjecture that the uniform distribution on order-n Latin squares is $(e^2 + o(1))/n$-spread is wide open as well. If true, these would imply:

- a uniformly random order-n Latin square a.a.s. no Latin subsquare of order four or more (recently proved by Allsop and Wanless [1]);
- the expected number of copies of the Fano plane in a uniformly random order-n Steiner triple system is $O(1)$.

In general, random designs are difficult to analyze, because we don't have sufficiently precise estimates for their number, or for the number of completions of a partial design. Hence, our approach to proving Theorem 1.6 is to instead *construct* a well-spread measure.

3. Refined absorption

For simplicity, we will restrict our attention to graphs in this section, but the definitions and theorems generalize naturally to hypergraphs.

3.1. Absorbers.

First, we need to extend the notion of the divisibility conditions in (†).

Definition 3.1. A graph G is K_q-*divisible* if

- $q - 1 \mid d_G(v)$ for every $v \in V(G)$ and
- $\binom{q}{2} \mid |E(G)|$.

Note that K_n is K_q-divisible if and only if (†) holds for $r = 2$.

Observation 3.2. *If G admits a K_q-decomposition, then G is K_q-divisible.*

The converse of the above observation does not always hold (for example, C_6 is K_3-divisible), but for large nearly complete graphs, it does.

Definition 3.3. A K_q-*absorber* for a graph L is a graph A such that $V(L) \subseteq V(A)$ is independent in A and both A and $A \cup L$ have a K_q-decomposition.

Example 3.4. See Figure 1.

If A is a K_q-absorber for a graph L, then since A has a K_q-decomposition, by Observation 3.2, A is K_q-divisible. After removing the edges of a K_q-divisible subgraph from a K_q-divisible graph, the result is also K_q-divisible. Hence, we have the following.

Observation 3.5. *If A is a K_q-absorber for a graph L, then both A and L are K_q-divisible.*

This observation tells us that for a graph, being K_q-divisible is a necessary condition for the existence of a K_q-absorber. It turns out that it is also sufficient.

Theorem 3.6. *For every graph L, if L is K_q-divisible, then a K_q-absorber for L exists.*

Theorem 3.6 was proved by Barber, Kühn, Lo, and Osthus [2]. The generalization to hypergraphs was proved by Glock, Kühn, Lo, and Osthus [8]. It also follows from appropriate strengthenings of the Existence Conjecture for designs, but it is a crucial ingredient in all known proofs of the Existence Conjecture, so that reasoning would be circular. Delcourt, Kelly, and Postle [3] recently provided a short proof.

3.2. **Omni-absorbers.** A K_q-absorber is analogous to the "absorbing gadgets" from the previous lecture; it can absorb a single "leftover" from our almost covering. However, there are many possibilities for what that leftover could be, and we don't know in advance what the leftover will be when constructing our absorber, so we need something stronger.

Definition 3.7. A *K_q-omni-absorber* for a graph X is a graph $\tilde{A}$ which is edge-disjoint from X such that for every K_q-divisible $L \subseteq X$, the graph $\tilde{A} \cup L$ has a K_q-decomposition.

The graph X should be viewed as analogous to the "flexible set" in the RMH introduced in the previous lecture.

Observation 3.8. *For every graph X, a K_q-omni-absorber for X exists.*

Proof. For each K_q-divisible $L \subseteq X$, let A_L be a K_q-absorber for L (which exists by Theorem 3.6), so that the sets $V(A_L) \setminus V(L)$ are pairwise disjoint. Let $\tilde{A}$ be the union of the A_L. □

The proof of the previous observation is similar to the idea from the previous lecture of using the Replacement Lemma to replace edges of our absorbing template with absorbing gadgets, so that the resulting hypergraph is robustly matchable. Here, we are working with a very inefficient template. The number of extra vertices used is exponential in $|E(X)|$.

Consequently, it is difficult to find such an omni-absorber unless X is very small. The "iterative absorption" proof of Glock, Kühn, Lo, and Osthus [8] essentially works in this way, using a sequence of cover down steps (hence the "iterative" in "iterative absorption") to shrink the leftover to live inside a constant-sized set, at which point it can be absorbed by an omni-absorber constructed as in the proof of Observation 3.8.

The proof would be much simpler if we had a more efficient way to construct omni-absorbers. This feat was accomplished by Delcourt and Postle [5].

Theorem 3.9 (Efficient omni-absorber theorem; Delcourt and Postle [5])**.** *For every $q \geq 3$, there exists $C \geq 1$ such that the following holds. If $X \subseteq K_n$ and $\Delta(X) \leq n/C$, then there exists a K_q-omni-absorber $\tilde{A} \subseteq K_n - X$ for X such that $\Delta(\tilde{A}) \leq C\Delta$, where $\Delta = \max\{\Delta(X), \sqrt{n}\log n\}$.*

The important point here is that, after adding the omni-absorber to X, the maximum degree essentially only increased by a constant factor. This fact is analogous to how, in the previous lecture, the flexible set of our RMH contained a linear proportion of the vertices. Since we are decomposing *edges* here rather than *vertices* as before, we want to compare the relative sizes of $|E(X)|$ and $|E(\tilde{A})|$ when thinking of efficiency.

Note: we will abuse notation and sometimes use X and $\tilde{A}$ to refer to the edge sets of these graphs.

3.3. **Omni-absorbers as robustly matchable hypergraphs.** Recall from the previous lecture that a hypergraph $\mathcal{R}$ is robustly matchable with respect to a flexible set U if, after deleting any admissible subset of U, the remaining hypergraph has a perfect matching. We can in fact make the connection between omni-absorbers and robustly matchable hypergraphs very explicit.

Definition 3.10. The *K_q-design hypergraph* of a graph G, denoted $\mathrm{Design}(G, K_q)$, is the hypergraph whose vertex set is $E(G)$ and whose hyperedges are the copies of K_q in G, i.e.

$$V(\Gamma) = E(G) \qquad \text{and} \qquad E(\Gamma) = \{E(Q) : Q \cong K_q,\ Q \subseteq G\}.$$

Observation 3.11. *The K_q-decompositions of a graph G are in bijection with the perfect matchings of* $\mathrm{Design}(G, K_q)$*.*

As we will see, omni-absorbers also correspond to robustly matchable subhypergraphs of the design hypergraph, but first we must specify a "decomposition family".

Definition 3.12. We say a K_q-omni-absorber $\tilde{A}$ for X has *decomposition family* $\mathcal{H}$ and *decomposition function* $\mathcal{Q}$ if $\mathcal{H}$ is a family of subgraphs of $X \cup \tilde{A}$ each isomorphic to K_q such that for every K_q-divisible subgraph L of X, there exists $\mathcal{Q}(L) \subseteq \mathcal{H}$ which is a K_q-decomposition of $L \cup \tilde{A}$.

Observation 3.13. *For $X \subseteq E(G)$, the K_q-omni-absorbers $\tilde{A} \subseteq G$ for X, together with a decomposition family, are in bijection with subhypergraphs of* $\mathrm{Design}(G, K_q)$ *that are robustly matchable with respect to flexible set X.*

The decomposition family $\mathcal{H}$ can be viewed as the edge set of the robustly matchable subhypergraph of $\mathrm{Design}(K_n, K_q)$, and $\tilde{A} \cup X$ (more specifically $E(\tilde{A}) \cup E(X)$) is its vertex set.

3.4. **Interlude: A proof of existence.** Assuming Theorem 3.9, let us discuss how to prove the existence conjecture (the proof works for all r) before turning to Theorem 1.6.

The proof is similar in spirit to the one discussed in the previous lecture, although a few steps require more sophisticated (but still relatively standard) probabilistic tools.

Step 1. Random reserve

Choose $X \subseteq E(K_n)$ by including each edge independently with probability $n^{-\varepsilon}$ for some small $\varepsilon > 0$.

Step 2. Construct the omni-absorber

Employ Theorem 3.9 to construct a K_q-omni-absorber $\tilde{A}$ for X.

Step 3. Almost cover

Find a K_q-packing of $K_n - (\tilde{A} \cup X)$. For this part, we can use hypergraph matching results applied to $\mathrm{Design}(K_n - (\tilde{A} \cup X), K_q)$ as a black box, but we need to do "regularity boosting" first.

Step 4. Complete cover down

Using edges from X, cover the remaining edges of $K_n - (\tilde{A} \cup X)$. This part can be done with the bipartite matching theorem from previous lecture. (This step and the previous one can also be done in one go with a black-box hypergraph matching result called "nibble with reserves".)

Step 5. Extend to complete decomposition using the omni-absorber

Find a K_q-decomposition of $\tilde{A}$ together with the remaining edges of X, using the fact that $\tilde{A}$ is a K_q-omni-absorber for X.

3.5. **So what makes it "refined"?** We have proved the existence without using the "refined" property of the omni-absorber yet. Let us first see the definition of this key property.

Definition 3.14. Let $\tilde{A}$ be an omni-absorber for X with decomposition family $\mathcal{H}$. For $C \geq 1$, we say $(\tilde{A}, \mathcal{H})$ is *C-refined* if $|\{H \in \mathcal{H} : e \in E(H)\}| \leq C$ for every edge $e \in X \cup \tilde{A}$.

Recall from the previous lecture that having constant maximum degree was essential for the success of using the robustly matchable hypergraph as a template for building an absorber. When viewing an omni-absorber $\tilde{A}$ with decomposition family $\mathcal{H}$, being C-refined is exactly this property when viewed as a robustly matchable subhypergraph of the design hypergraph.

Observation 3.15. *For an omni-absorber $\tilde{A}$ with decomposition family $\mathcal{H}$, we have that $(\tilde{A}, \mathcal{H})$ is C-refined if and only if the maximum degree of the corresponding robustly matchable subhypergraph is at most C.*

The main result of Delcourt and Postle is not just that efficient omni-absorbers exist, but that C-refined ones exist.

Theorem 3.16 (Refined omni-absorber theorem; Delcourt and Postle [5])**.** *In the Efficient omni-absorber theorem, the omni-absorber is also C-refined.*

This theorem is nontrivial even for $r = 1$:

- $\emptyset$ is an omni-absorber for X, but the decomposition family contains every q-set, so it is not refined;
- the template lemma from the previous lecture is the solution!
- the proof goes by induction on r, and the template lemma is the base case.

To reiterate:

- Observation 3.13 tells us that omni-absorbers, together with a decomposition family, can be viewed as robustly matchable hypergraphs, which can be either used as an absorber or as a template for building an absorber;
- the Efficient omni-absorber theorem (Theorem 3.9) yields omni-absorbers whose corresponding robustly matchable subhypergraph has a linear-sized flexible set, which is key for using it as an absorber;
- the Refined omni-absorber theorem (Theorem 3.16) additionally provides omni-absorbers whose corresponding robustly matchable subhypergraphs have constant maximum degree (by Observation 3.15), which is key for using it as a template.

Remark 1. Delcourt and Postle [5] understood that proving the refined omni-absorber theorem would yield new applications by using the omni-absorber as a template. However, proving the stronger refined omni-absorber theorem is also key for their proof strategy; their proof uses induction on the uniformity r, and having the stronger refined property on the omni-absorbers obtained inductively is important.

4. Boosters

A refined omni-absorber can be used as a deterministic template. To make the distribution spread, we replace each predetermined clique by a randomly chosen private gadget called a booster.

Definition 4.1. Let $q > 1$ be an integer.

- A K_q*-booster* is a graph B along with two K_q-decompositions $\mathcal{B}_1$ and $\mathcal{B}_2$ of B such that $\mathcal{B}_1 \cap \mathcal{B}_2 = \varnothing$.
- A tuple $(B, \mathcal{B}_1, \mathcal{B}_2, S)$ is a *rooted* K_q*-booster* if
  - B is a graph,
  - $\mathcal{B}_2$ is a K_q-decomposition of B, called the *off-decomposition*,
  - $S \cong K_q$, $V(S) \subseteq V(B)$, and S is edge-disjoint from B,
  - $\mathcal{B}_1$ is a K_q-decomposition of $B \cup S$ with $S \notin \mathcal{B}_1$, called the *on-decomposition*, and
  - $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$.

Figure 1 also shows a booster. A booster can be thought of as an L-absorber when $L \cong K_q$.

Now suppose $\tilde{A}$ is a K_q-omni-absorber for X with decomposition family $\mathcal{H}$ and decomposition function $\mathcal{Q}$, and suppose $S \in \mathcal{H}$. If we find a rooted booster $(B, \mathcal{B}_1, \mathcal{B}_2, S)$ whose non-root edges are disjoint from $\tilde{A} \cup X$, then we may replace S in the template by the booster. Indeed, for each K_q-divisible leftover $L \subseteq X$, the old decomposition $\mathcal{Q}(L)$ either uses S or does not use S.

- If $S \in \mathcal{Q}(L)$, replace S by the on-decomposition $\mathcal{B}_1$.
- If $S \notin \mathcal{Q}(L)$, add the off-decomposition $\mathcal{B}_2$.

The resulting graph is again a K_q-omni-absorber for X.

Lemma 4.2 (Booster replacement lemma)**.** *If $\tilde{A}$ is a K_q-omni-absorber for X, and one clique in its decomposition family is replaced by a private rooted booster, then the resulting graph is still a K_q-omni-absorber for X.*

This is exactly the replacement lemma from the previous lecture, taking place in the design hypergraph. However, in our setting the replacement will be chosen randomly, and the purpose is not just absorption but to construct a spread distribution.

The bottleneck is the rooted density of the booster. Roughly, if a booster has many non-root vertices compared with the number of forced cliques, then there are many ways to embed it, and random embeddings are spread.

Definition 4.3. Let $q > 1$.

- For a nonempty K_q-packing $\mathcal{H}$ of a graph G and $R \subseteq V(G)$, the *rooted density* of $\mathcal{H}$ and R is
$$d(\mathcal{H}, R) \coloneqq |\mathcal{H}| \Bigg/ \left| \bigcup_{H \in \mathcal{H}} V(H) \setminus R \right|,$$
and the *maximum rooted density* of $\mathcal{H}$ and R is
$$m(\mathcal{H}, R) \coloneqq \max_{\mathcal{H}' \subseteq \mathcal{H}} d(\mathcal{H}', R).$$
- For a rooted K_q-booster $(B, \mathcal{B}_1, \mathcal{B}_2, S)$, the *rooted density* is $\max_{\mathcal{H} \in \{\mathcal{B}_1, \mathcal{B}_2\}} m(\mathcal{H}, V(S))$.

We can find boosters with rooted density at most $2/(q-2)$, which is what leads to the exponent $n^{-(q-6)/2}$ in the final threshold bound.

Remark 2. Even without Theorem 2.2, we could get a nontrivial upper bound on the threshold using the refined omni-absorber as a template. Note that the rooted density of a K_q-booster essentially determines the expected number of rooted copies of that booster in $\mathcal{H}^{(q)}_{n,p}$. However, since we would need both the on and off decomposition to be in $\mathcal{H}^{(q)}_{n,p}$, the best bound that we could prove with this approach would have a "4" in the denominator of the exponent rather than a "2".

4.1. **The conditional Lovász Local Lemma distribution.** We can prove that spread boosters may be embedded edge-disjointly in a manner almost as spread as if they were embedded independently at random. To that end, we require the following variant of the Lovász Local Lemma [6] for the conditional distribution (see Haeupler, Saha, and Srinivasan [9] and Jain and Pham [10]).

Lemma 4.4. *Let $\{X_i\}_{i \in I}$ be independent random variables, and let $\{\mathcal{E}_j\}_{j \in J}$ be events where each $\mathcal{E}_j$ depends on a subset $S_j \subseteq I$ of the variables. Let Γ be a graph with vertex set J where $S_j \cap S_{j'} = \emptyset$ for all distinct non-adjacent $j, j' \in J$. Let $\mathbb{P}$ be the product measure on the random variables $\{X_i\}_{i \in I}$, and let $\mathcal{E}$ be an event depending on a subset $S \subseteq I$. If $\mathbb{P}[\mathcal{E}_j] \leq p$ for all $j \in J$ and $4p\Delta(\Gamma) \leq 1$, then*
$$\mathbb{P}\left[\mathcal{E} \,\middle|\, \bigcap_{j \in J} \overline{\mathcal{E}_j}\right] \leq \mathbb{P}[\mathcal{E}] \exp(6pN),$$
where N is the number of events $\mathcal{E}_j$ with $S_j \cap S \neq \emptyset$.

Recall the bipartite hypergraph matching lemma from the previous lecture, which was used to determine which absorbing gadgets would be used to replace which edges of the template. Lemma 4.4 (or at least, the idea of its proof) can be used in a similar way to determine which cliques in the decomposition family of our omni-absorber are replaced by which boosters.

As a simplified version of this argument, we instead focus on a "spread version" of the bipartite hypergraph matching lemma.

Lemma 4.5 (Bipartite hypergraph matching lemma)**.** *For every $k \geq 2$, there exists a constant $C \geq 1$ such that the following holds. If $\mathcal{G}$ is a bipartite k-uniform hypergraph with bipartition (A, B), and if there exists $D \geq 1$ such that*

- *every vertex of A has degree at least $8kD$ in $\mathcal{G}$ and*
- *every vertex of B has degree at most D in $\mathcal{G}$,*

then there is a (C/D)-spread distribution on A-perfect matchings of $\mathcal{G}$.

Exercise 4.6. *Prove Lemma 4.5.*

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