EXCILL V: EXTREMAL COMBINATORICS AT ILLINOIS
SUMMER SCHOOL ON DESIGN THEORY
LECTURE 2. INTRODUCTION TO THE ABSORPTION METHOD

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1. Introduction

This lecture is devoted to the proof of the following theorem. Here, $\mathcal{H}^{(k)}_{n,p}$ denotes the random k-uniform n-vertex hypergraph in which every edge is included independently with probability p.

Theorem 1.1. *If $p \geq 1/n$, then a.a.s. $\mathcal{H}^{(3)}_{n,p}$ has a perfect matching, provided $3 \mid n$.*

Exercise 1.2. *While going through the proof, or after:*

- *generalize to $\mathcal{H}^{(k)}_{n,p}$ where $p \geq 1/n^{k/2-\gamma}$;*
- *explain why the proof fails for $p \leq 1/n^{k/2+\gamma}$.*

Why prove this theorem?

1. Pedagogically, it is one of the most accessible applications of the distributive absorption / absorber-template method.
2. In many ways, $\mathcal{H}^{(3)}_{n,p}$ shares properties with a uniformly random order-n Steiner triple system; for example, the expected number of edges is essentially the same, and the expected value of each codegree = 1. Hence, it is a natural precursor to Kwan's result [11] that almost all Steiner triple systems have a perfect matching and also to Montgomery's results [13] on the Ryser–Brualdi–Stein conjecture and Brouwer's conjecture.
3. Recall that perfect matchings are $(n, k, 1)$-Steiner systems. This result can be viewed as a 1-dimensional version of the general thresholds for designs conjecture [7, Conjecture 1.14], and the proof is a good starting point for learning how refined absorption can be used to bound the threshold in the general case [1].

Where does the proof come from? Essentially all of the ideas for Theorem 1.1 are present in Kwan's [11] proof that almost all Steiner triple systems have a perfect matching. For Exercise 1.2, the ideas can be found in the paper of Ferber and Kwan [3]. Both papers prove stronger results so have additional complications that we will not go into.

Remark 1.

- The problem of determining this threshold was known as Shamir's Problem. The actual threshold is $\log n/n^{k-1}$. The lower bound can be proved easily by observing that the threshold is at least the threshold for $\mathcal{H}^{(k)}_{n,p}$ to contain no isolated vertices. The upper bound was proved by Johansson, Kahn, and Vu [6] in 2008, and it also follows from the Park–Pham Theorem [14] (or the fractional version [4]). However, both proofs are non-constructive, whereas the absorption-based proof is constructive, so it would be interesting to find a constructive proof for any $p < 1/n^{k/2}$.
- Arguably the first paper using what we would now call an absorption approach is the paper of Krivelevich [10] on the threshold for $G(n, p)$ to have a triangle-factor, which is closely related to the threshold for a 3-uniform binomial random hypergraph to have a perfect matching.

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- The fact that for $k = 2$ it matches the actual threshold (up to the n^γ factor) is significant – this partially explains why Montgomery's proof [12] on the threshold for bounded-degree spanning trees in $G(n, p)$ works.

2. Almost covering

First, observe the following.

Lemma 2.1. *If $p \geq 1/n$, then a.a.s. every set of at least $n^{2/3}$ vertices of $\mathcal{H}^{(3)}_{n,p}$ contains an edge.*

This lemma implies that a.a.s. there is a matching covering all but $n^{2/3}$ vertices, which we can construct greedily.

Before constructing the matching, we want to build an "absorbing structure" to deal with those last $n^{2/3}$ vertices.

Remark 2. In general, absorption is an approach by which we can potentially turn an "almost packing/covering" result into a "perfect packing/covering" result. In many absorption proofs, the almost covering part is done by a nibble or random greedy strategy, for which we can often use hypergraph matching results as a blackbox (see [8, 9]). Here, that is not necessary.

3. Absorption strategy

Here is how we will use absorption to prove the result.

Definition 3.1. A k-uniform hypergraph $\mathcal{H}$ is *robustly matchable* with respect to a *flexible set* U if, for every $X \subseteq U$ such that $k \mid |X|$, there is a perfect matching of $\mathcal{H} - X$.

We will abbreviate *robustly matchable hypergraph* as RMH.

Example 3.2. Let $\mathcal{H}$ be an n-vertex k-uniform hypergraph.
- If $\mathcal{H}$ is just a perfect matching (i.e. it's n/k pairwise disjoint edges), then it is robustly matchable with respect to $\emptyset$.
- If $\mathcal{H}$ is complete, then it is robustly matchable with respect to $V(\mathcal{H})$.

Observation 3.3. *If $\mathcal{H}$ is robustly matchable, then $\mathcal{H}$ has a perfect matching.*

The goal is to find a robustly matchable hypergraph $\mathcal{R}$ inside $\mathcal{H}^{(3)}_{n,1/n}$ with a linear-sized flexible set, which we will use as an "absorber". We will be able to greedily find a matching covering almost all of the vertices outside $\mathcal{R}$, then cover the remaining vertices with edges containing vertices from $\mathcal{R}$'s flexible set. Finally, we can complete the matching using the robust matchability property of $\mathcal{R}$.

By Observation 3.3, we cannot really fix the vertices of $\mathcal{R}$, or else we have not made the problem any easier. Instead, we'll fix some set of vertices to be the flexible set, and use a small portion of the remaining vertices to build the robustly matchable hypergraph.

The two main lemmas we need are the following.

Lemma 3.4 (Absorber lemma)**.** *Let $\alpha \ll 1$, and let $\mathcal{H} \sim \mathcal{H}^{(3)}_{n,p}$. If $p \geq 1/n$ and $U \subseteq V(\mathcal{H})$ has size at most αn, then a.a.s. $\mathcal{H}$ contains a robustly matchable subhypergraph with flexible set U.*

Lemma 3.4 is where the key ideas are, and we will prove it in Section 4. The next lemma requires standard probabilistic techniques.

Lemma 3.5 (Cover down lemma)**.** *Let $\alpha > 0$, and let $\mathcal{H} \sim \mathcal{H}^{(3)}_{n,p}$. If $p \geq 1/n$ and $U \subseteq V(\mathcal{H})$ is a set of at least αn vertices, then a.a.s. $\mathcal{H}$ satisfies the following. For every $B \subseteq V(\mathcal{H}) \setminus U$, there is a matching covering every vertex outside of $U \cup B$, which is disjoint from B.*

Proof sketch. First, show that a.a.s.

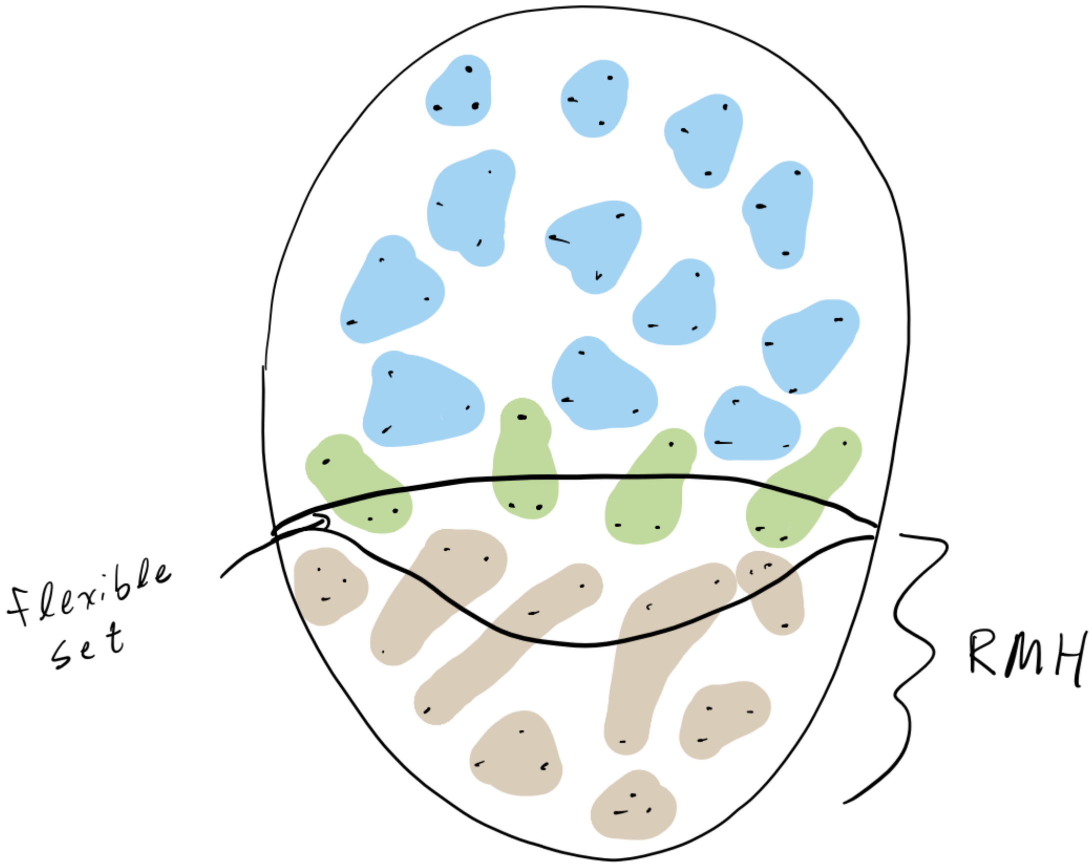


FIGURE 1. Cover down and absorption

- every subset of size at least $n^{2/3}$ contains an edge in $\mathcal{H}$
- for every set $X \subseteq U$ of size at most $2n^{2/3}$, every vertex is contained in an edge where both of the other two vertices are in $U \setminus X$.

The first property means that we can construct a matching greedily that covers all but at most $n^{2/3}$ vertices not in U or B, and the second property allows us to extend it greedily to a matching covering the remaining vertices, while using at most $2n^{2/3}$ additional vertices from U. □

Assuming the previous two lemmas, we can easily prove our result. Also, see Figure 1.

Proof of Theorem 1.1. Let $\alpha \ll 1$, let $p \geq 1/n$, and let $\mathcal{H} \sim \mathcal{H}^{(3)}_{n,p}$. Choose $U \subseteq V(\mathcal{H})$ of size αn arbitrarily. By Lemma 3.4, $\mathcal{H}$ contains a robustly matchable subhypergraph $\mathcal{R}$ with flexible set U. By Lemma 3.5 (with $V(\mathcal{R}) \setminus U$ playing the role of B), $\mathcal{H}$ contains a matching M_1 covering every vertex outside of $V(\mathcal{R})$, which is disjoint from $V(\mathcal{R}) \setminus U$. Let $X = V(M_1) \cap U$, and observe that $3 \mid |X|$. Since $\mathcal{R}$ is robustly matchable with respect to U, the hypergraph $\mathcal{R} - X$ has a perfect matching, which we call M_2. Now $M_1 \cup M_2$ is a perfect matching of H, as desired. □

Remark 3. The set B presents no problem in Lemma 3.5 in this case. In more complicated examples involving a nibble result, one would need to do a "regularity boosting" step. The reason is that the size of B affects how regular the hypergraph is, which in turn affects the size of the leftover (sometimes called the "leave"). If $|B| \approx \beta n$, then the leftover we'd get would be at most εn if $\beta \ll \varepsilon$. But as we'll see in the proof of the absorber lemma, we'll have $\alpha \ll \beta$, and we'd need $\varepsilon < \alpha$ for the cover down to work.

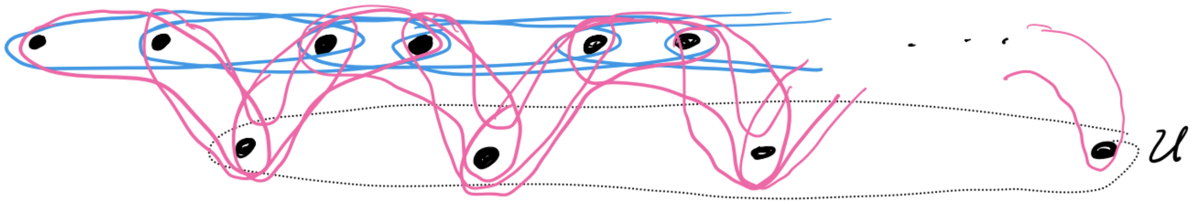


FIGURE 2. The robustly matchable hypergraph we will use as a template

4. Proof of the absorber lemma

It remains to prove the absorber lemma. We will use the "distributive absorption" or "absorber-template method" pioneered by Montgomery. First, we need the template lemma.

Lemma 4.1 (Template lemma)**.** *For every k and every n divisible by k, there exists a k-uniform n-vertex hypergraph which is robustly matchable with respect to a flexible set of size at least n/k with maximum degree at most $2k$.*

Proof sketch. Let U be a subset of n/k vertices. Imagine ordering the vertices of $\mathcal{H}$ as

$$v_1, \dots, v_{k-1}, u_1, v_k, \dots, v_{2k-1}, u_2, v_{2k}, \dots, v_{3k-1}, u_3, \dots, u_{n/k}.$$

The edge set of $\mathcal{H}$ will be the union of two tight paths, one on $V(\mathcal{H}) - U$ and the other on $V(\mathcal{H})$ (using this vertex ordering). See Figure 2.

Note that after deleting any subset of U, the remainder of $\mathcal{H}$ has a tight Hamilton path, and a tight Hamilton path where the uniformity divides the number of vertices contains a perfect matching. □

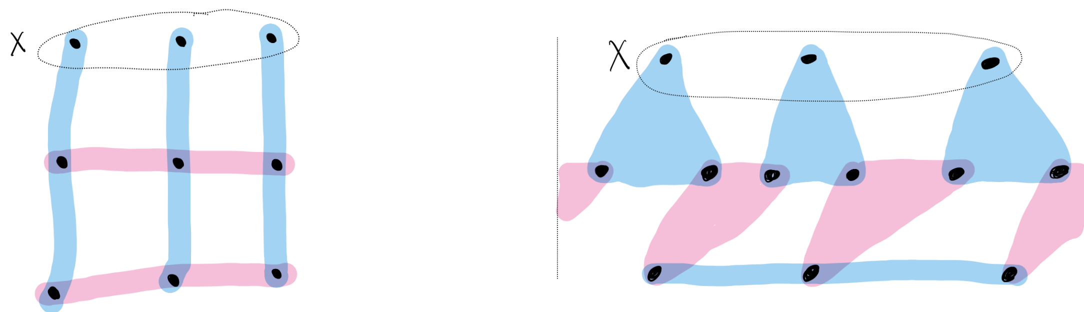
Remark 4. The absorber-template approach was first used by Montgomery (see [12, Lemma 3.43]). Instead of a robustly matchable hypergraph, he used a *robustly matchable bipartite graph* (RMBG, or Richard Montgomery Bipartite Graph). There, the flexible set is a linear-sized subset of one of the two parts, and the robust matchability property is that there is a perfect matching after deleting any subset of the flexible set that leaves the two parts balanced. Montgomery's proof uses a constant number of independent random perfect matchings. Kwan [11, Lemma 5.2] observed that an RMBG can be used to construct a "robustly matchable tripartite hypergraph". We could also use this structure, but it is slightly less convenient, since we must delete a prescribed number of vertices from the flexible set in order to guarantee a perfect matching. Ferber and Kwan [3, Lemma 7.3] used a random construction to prove a variant of Lemma 4.1, but the proof provided here is more elementary. Although the bipartite graph construction is less convenient in this setting, in some applications of the absorber-template method the partite structure is necessary.

We also note that the hypergraph constructed in the proof of Lemma 4.1 is very similar to the absorber constructed by Rödl, Ruciński, and Szemerédi [16] in one of their earliest works using the absorption method. That this construction could be used as a "template" was first observed by Delcourt and Postle [2].

Definition 4.2. We say a hypergraph $\mathcal{A}$ is an X-*absorbing gadget* for $X \subseteq V(\mathcal{A})$ if both $\mathcal{A}$ and $\mathcal{A} - X$ have perfect matchings and each edge of $\mathcal{A}$ intersects X in at most one vertex. We say the vertices in X are the *roots* and the other vertices are *non-roots*.

Example 4.3. See Figure 3

Using a robustly matchable hypergraph as a template, we replace edges with absorbing gadgets to build an absorber inside our host structure. The following simple lemma states that replacing



(A) A 9-vertex absorbing gadget

(B) A 12-vertex absorbing gadget

FIGURE 3. Absorbing gadgets

an edge e in a robustly matchable hypergraph with an e-absorbing gadget preserves the robust matchability property.

Lemma 4.4 (Replacement lemma)**.** *If $\mathcal{R}$ is robustly matchable with respect to flexible set U, and if $\mathcal{R}'$ is obtained from $\mathcal{R}$ by replacing an edge e with an e-absorbing gadget whose non-root vertices are not in $\mathcal{R}$, then $\mathcal{R}'$ is also robustly matchable with respect to U.*

Proof sketch. For any $X \subseteq U$ such that $k \mid |X|$, we have a perfect matching of $\mathcal{R} - X$. If e is in this perfect matching, we remove it from the matching and combine it with the perfect matching of our absorbing gadget to obtain a perfect matching of $\mathcal{R}'$. If e is not in this perfect matching, then we keep it in the matching and add the perfect matching of our absorbing gadget minus e to obtain a perfect matching of $\mathcal{R}'$. □

The next lemma concerns the distribution of absorbing gadgets inside our random hypergraph, which we will use to build our robustly matchable hypergraph.

Lemma 4.5 (Gadget lemma)**.** *Let $1/n \ll \beta, 1/C \ll 1/\Delta \leq 1$, let $p \geq 1/n$, and let $\mathcal{H} \sim \mathcal{H}^{(3)}_{n,p}$. If $\mathcal{T}$ is a 3-uniform hypergraph with $V(T) \subseteq V(\mathcal{H})$ such that*

$$|V(\mathcal{T})| \leq \beta n \qquad \text{and} \qquad \Delta(\mathcal{T}) \leq \Delta,$$

then a.a.s. the following holds for some $\mu > 0$.

- *For every $e \in E(\mathcal{T})$, there are at least μ/C e-absorbing gadgets in $\mathcal{H}$ whose non-root vertices lie in $V(\mathcal{H}) \setminus V(\mathcal{T})$.*
- *For every $v \in V(\mathcal{H}) \setminus V(\mathcal{T})$, there are at most $C\Delta\beta\mu$ e-absorbing gadgets in $\mathcal{H}$ containing v for $e \in E(\mathcal{T})$.*

Proof sketch. Let $\{x, y, z\} \in E(\mathcal{T})$. Count the number of $\{x, y, z\}$-absorbing gadgets isomorphic to the one in Figure 3b. The gadget has 9 "non-root" vertices and 7 edges, so the expected number of these $\{x, y, z\}$-absorbing gadgets in $\mathcal{H}$ is on the order of n^9p^7, and for any fixed $w \notin \{x, y, z\}$, the number of them containing w is at most n^8p^7. So let $\mu = n^9p^7$. By a concentration inequality (e.g. Kim–Vu), the number of these gadgets is close to the expected number with high probability and we can union bound over all choices of $e \in E(\mathcal{T})$ and $v \in V(\mathcal{H}) \setminus V(\mathcal{T})$. □

In the gadget lemma, one should think of the number of non-root vertices in the gadget as "degrees of freedom". If we fix an additional non-root vertex, then the expected number of gadgets using that vertex decreases by a factor of n. Given that $|E(\mathcal{T})| \leq \Delta\beta n \ll n$ and each gadget has a constant number of non-root vertices, this inequality suggests that we should be able to replace each edge e of the template $\mathcal{T}$ with an e-absorbing gadget in $\mathcal{H}$, one-by-one in a greedy fashion, so that each gadget selected is vertex-disjoint from $V(\mathcal{T})$ and all previously selected gadgets. By

Lemma 4.4, the resulting hypergraph would be robustly matchable, which would complete the proof of Lemma 3.4.

This approach indeed works for $p \geq 1/n$, but in other settings (such as in Exercise 1.2), we may need to choose the absorbing gadgets more carefully. To that end, we introduce the following definition and lemma.

Definition 4.6. A hypergraph $\mathcal{G}$ is *bipartite with bipartition* (A, B) if A and B are disjoint sets satisfying $V(\mathcal{G}) = A \cup B$ and every edge of $\mathcal{G}$ contains exactly one vertex from A. A matching of $\mathcal{G}$ is *A-perfect* if every vertex of A is in an edge of the matching.

Lemma 4.7 (Bipartite hypergraph matching lemma)**.** *Let $\mathcal{G}$ be a bipartite k-uniform hypergraph with bipartition (A, B). If there exists $D \geq 1$ such that*

- *every vertex of A has degree at least $8kD$ in $\mathcal{G}$ and*
- *every vertex of B has degree at most D in $\mathcal{G}$,*

then $\mathcal{G}$ contains an A-perfect matching.

Proof. The lemma can be deduced from a result of Haxell [5] on independent transversals (proved with a topological approach) or of Reed [15] (proved with the Lovász Local Lemma). □

Combining the above, we can prove the absorber lemma.

Proof sketch of Lemma 3.4. Let $1/n \ll \alpha \ll \beta \ll 1/C \ll 1/\Delta \ll 1$, and let $\mathcal{H} \sim \mathcal{H}^{(3)}_{n,p}$. By Lemma 4.1, we can consider a robustly matchable hypergraph $\mathcal{T}$ on a subset of at most βn vertices of $\mathcal{H}^{(3)}_{n,1/n}$ with maximum degree at most Δ and flexible set U of size αn to use as a template.

Let $\mathcal{G}$ be the bipartite hypergraph with bipartition $(E(\mathcal{T}), V(\mathcal{H}) \setminus V(\mathcal{T}))$, where we include the edge $\{e\} \cup (V(\mathcal{A}) \setminus e)$ for every e-absorbing gadget $\mathcal{A}$ (with non-root vertices not in $V(\mathcal{T})$), where $e \in E(\mathcal{T})$. By Lemma 4.5 and 4.7, $\mathcal{G}$ has an $E(\mathcal{T})$-perfect matching, which corresponds to a set $\{\mathcal{A}_e : e \in E(\mathcal{T})\}$ where $\mathcal{A}_e \subseteq \mathcal{H}$ is an e-absorbing gadget and the sets of non-root vertices of these absorbing gadgets are pairwise disjoint.

By Lemma 4.4, the hypergraph obtained by replacing each $e \in E(\mathcal{T})$ with $\mathcal{A}_e$ is also robustly matchable with the same flexible set U, which completes the proof. □

5. Hints for Exercise 1.2

- Lemma 4.7 is useful in extending the proof of the Cover Down Lemma.
- Observe that, if $\mathcal{A}$ is an X-absorbing gadget where $|X| = k$, then the hypergraph with X added as an edge is the union of two perfect matchings, and its line graph is a k-regular bipartite graph.
- It is helpful to use the sprinkling technique. That is, let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 \sim \mathcal{H}^{(k)}_{n,p/4}$, and observe that $\mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$ is stochastically dominated by $\mathcal{H}^{(k)}_{n,p}$. Use $\mathcal{H}_1$ in the absorber lemma, use $\mathcal{H}_2$ to find the majority of the matching in the cover down lemma, and use $\mathcal{H}_3$ to find the edges between the leftover and the flexible set.

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